A nonlinear drift which leads to κ-generalized distributions

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Non-Gaussian distributions are frequently observed in a variety of systems such as physical, chemical, economical or social systems. Examples of non-Gaussian distributions are Lévy stable distributions, and Tsallis q-generalized distributions in the non-extensive statistical mechanics based on Tsallis' entropy [1]. A common key feature of them is the presence of an asymptotic power-law tail.
There is another type of non-Gaussian distributions with asymptotic power-law tails, which is called a κ-generalized distribution. It has been studied originally in the context of statistical physics by Kaniadakis [2]. Maximizing Kaniadakis' κ-entropy under appropriate constraints leads to a κ-generalized Gaussian distribution, which can be written as

$$g_\kappa(x) \propto \exp_\kappa\left(-\beta x^2\right). \tag{1}$$

Here κ is a real deformed parameter ($-1 < \kappa < 1$), β is a constant, and $\exp_\kappa(x)$ is κ-exponential function defined by

$$\exp_\kappa(x) = \left(\kappa x + \sqrt{1+\kappa^2 x^2}\right)^{\frac{1}{\kappa}}. \tag{2}$$

This $\exp_\kappa(x)$ reduces to the standard $\exp(x)$ in the limit of $\kappa = 0$. Note that for a large value of x, the κ-exponential function obeys a power-law as $\exp_\kappa(x) \sim x^{1/\kappa}$, whereas for a small value of x it approximately behaves as $\exp(x)$. The κ-generalized distributions have been shown to well explain, for example, the energy distributions of cosmic rays [2], and the size distribution of personal incomes [3]. We studied the asymptotic behavior of the κ-generalized nonlinear Fokker-Planck(FP) equation, which stationary solution is a κ-generalized Gaussian distribution [4]. Furthermore a κ-generalized Gaussian is also derived by generalizing Gauss' law of error [5].

On the other hand, Lutz [6] has recently shown the connection between anomalous transport in an optical lattice and Tsallis q-generalized distributions based on a linear FP equation with a nonlinear drift coefficient,

$$K^{\mathrm{ol}}(p) = -\frac{\alpha p}{1+\left(\frac{p}{p_c}\right)^2}. \tag{3}$$

The drift $K^{\mathrm{ol}}(p)$ represents a capture force with damping coefficient α, and this force acts only on slow particles whose momentum is smaller than the capture momentum p_c. A characteristic feature of this nonlinear drift is that: for a small momentum $|p| < p_c$, the drift is approximately linear $K^{\mathrm{ol}}(p) \sim -p$, i.e., it reduces to a familiar Ornstein-Uhlenbeck process; whereas for a large momentum $|p| > p_c$, it asymptotically decreases as $K^{\mathrm{ol}}(p) \sim -1/p$. In contrast to most systems with power-law distributions which are often described by nonlinear kinetic equations [7], the above process is described by an ordinary linear FP equation. Consequently standard methods can be applied to analyze the problem. Lutz also pointed out an explicit correspondence between ergodicity breaking in a system described by power-law tail distributions and the divergences of the moments of these distributions. It is worth stressing that the Lutz analysis is not restricted to anomalous transport in an optical lattice, but can be applied to a wide class of systems described by a FP equation with a drift coefficient decaying asymptotically as $-1/p$.

In this contribution we consider an ordinary linear FP equation

$$\frac{\partial}{\partial t} w(p,t) = -\frac{\partial}{\partial p}\Big(K(p)\, w(p,t)\Big) + D\frac{\partial^2}{\partial p^2} w(p,t), \tag{4}$$

with another type of momentum-dependent drift coefficient,

$$K(p) = -\frac{\alpha p}{\sqrt{1+\left(\frac{p}{p_c}\right)^4}}, \tag{5}$$

and a constant diffusion coefficient D. Note that this drift coefficient $K(p)$ also asymptotically decreases as $-1/p$ for a large momentum $|p| > p_c$.
It is shown that the stationary solution of this FP equation (4) with the nonlinear drift coefficient (5) is nothing but a κ-generalized Gaussian distribution (1). Similar to a q-distribution in the Lutz analysis, the deformed parameter κ can be expressed in terms of the microscopic parameters as

$$\kappa = \frac{2D}{\alpha p_c^2}, \tag{6}$$

which allow us to give a physical interpretation of the κ-distribution.

Keywords
non-Gaussian distribution, κ-generalized distribution, anomalous diffusion, non-linearity, asymptotic power-law

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