STUDY OF STOCK EXCHANGE INDEX DYNAMICS BY USING THE FRACTIONAL MARKET MODEL

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We consider current speculative bubbles and crashes of indices (which form *cusp-like* local peaks superposed with oscillations) on different stock markets. We found that the rising and falling paths of the peaks are well described by a generalized exponential function or the Mittag-Leffler (ML) one superposed with various types of oscillations. We found that the ML function superposed with oscillations is a solution of the nonhomogeneous fractional relaxation equation (applied independently to each path):

$$\frac{dX(y)}{dy} = -(\tau_1)^{-\alpha}\, {}_0D_y^{1-\alpha} X(y) + \frac{1}{G_0}\,(\tau_0)^{-\alpha}\, {}_0D_y^{1-\alpha} U(y) + \frac{1}{G_0}\frac{d\,U(y)}{dy}, \tag{1}$$

which defines here our Fractional Market Model (FMM) [1] of index dynamics. This model can be also called the Rheological Model of Market since Eq.(1) is an analog of the one which describes the slowing down relaxation of viscoelastic materials [2-4] (then the index X is an analog of the material strain and the excess demand or supply U is that of the applied stress). The notation used is as follows: ${}_0D_y^{1-\alpha}X(y)$ is the Riemann-Liouville fractional differentiation [3,4] of the order $1-\alpha > 0$ (where $\alpha > 0$) and the variable $y =| t_{MAX} - t|$ is the time lag from the moment t_{MAX} when X assumes its maximum. Here, the relaxation times τ_0, τ_1 and coefficient G_0 are time-independent and the inhomogeneity $U(y) \sim \cos(\omega\, y)\cos(\Delta\omega\, y)$, where the frequencies $\Delta\omega < \omega \ll 1$. An Eq.(1) can be also considered as a generalisation of the well known in economy relation between temporary excess demand or supply and the temporary price or index change per unit time.
The approximate solution of Eq.(1) has the form

$$X(y) \approx \left[X(0) - \frac{U(0)}{G_0}\left(\frac{\tau_1}{\tau_0}\right)^{\alpha}\right] E_{\alpha}\left(-\left(\frac{y}{\tau_1}\right)^{\alpha}\right) + \frac{U(0)}{G_0}\left(\frac{\tau_1}{\tau_0}\right)^{\alpha}\cos(\omega\, y)\cos(\Delta\omega\, y), \tag{2}$$

where $E_\alpha(\ldots)$ is the Mittag-Leffler (ML) function (cf. the monotonic curves shown in Fig. 1). The ML function is the solution of the free relaxation equation defined by the homogeneous part of Eq.(1). Indeed, solution (1) was fitted to empirical data concerning recent hossa and bessa on Dow Jones (cf. oscillating curves shown in Fig.1).
In Table 1 the values of the main parameters defining solution (1) are shown. As it is seen,

Table 1: Main parameters definig solution (1)

Path	α	τ_1 $[td]$	t_{MAX} $[td]$	ω $[td^{-1}]$	$\Delta\omega$ $[td^{-1}]$
Raising	0.69	270.13	660	0.020	0.019
Falling	0.50	400.0	625	0.11	0.047

the values of exponent α for both paths are distinctly smaller than 1 and the relaxation time for the falling path τ_1 estimates the duration time of present financial crisis. Note that for

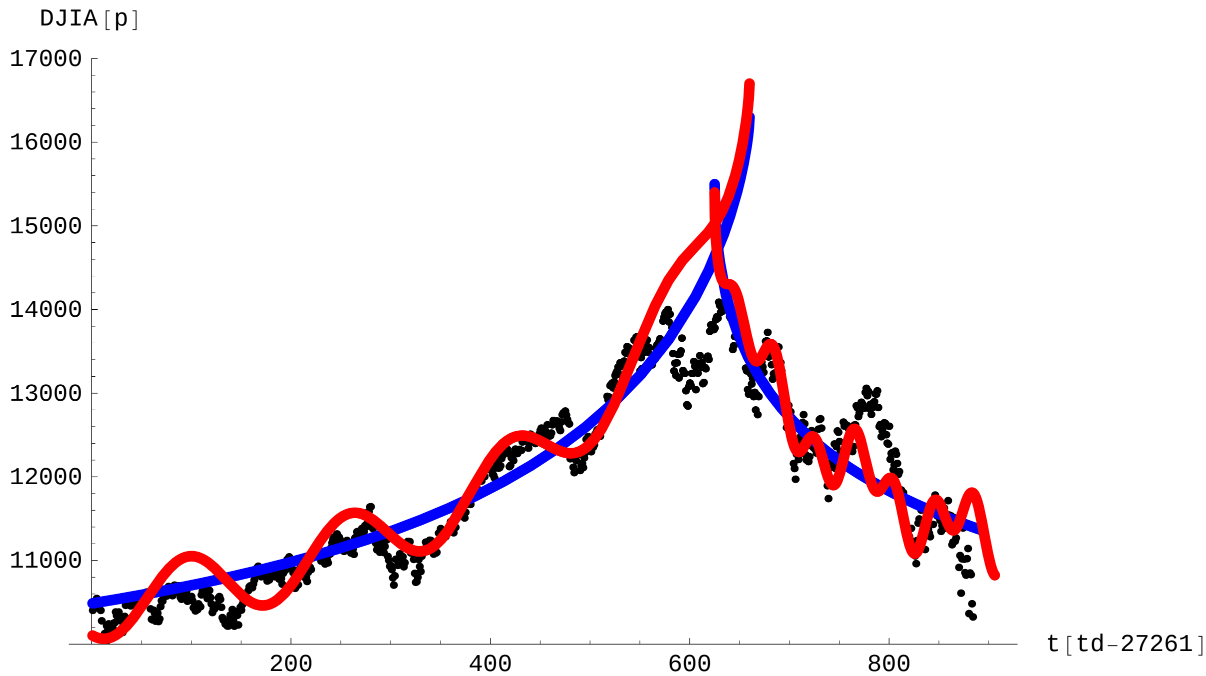

Figure 1: The index Dow Jones (black dots) for 894 trading days (i.e., from 27261 trading day counting from the index beginning until 28155 trading day) from 2005.04.01 until 2008.10.20. The oscillating curves are predicted by solution (2) while the monotonic curves are the predictions of the Mittag-Leffler function.

$t \to t_{MAX}$ the value of return per unit time $\frac{dX(y)}{dy} \sim \frac{1}{|t_{MAX}-t|^{1-\alpha}}$, which can be the "finger print" of some dynamical phase transition from the bullish to bearish state of the stock market. However, the basic dilemma in rationalizing relaxation curves is still the fact that for any set of data a number of the relaxation mechanisms can be made responsible for the observed behaviour.

Keywords
fractional relaxation equation, Mittag-Leffler function, stock exchange index, crash, viscoelastic material

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