Limiting eigenvalue distributions of random matrices

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A random matrix is a matrix whose elements are random variables, or equivalently, a matrix-valued random variable. The history of random matrices can be traced back to multivariate statistics and mathematical physics in early-mid 20th-century, and a significant amount of knowledge has been accumulated since then. In recent years several new applications of random matrices in the field of informatics have emerged. They include statistical learning, data science, mathematical finance [1], and wireless communication [2], to mention a few. In these research fields, matrices of the sample-covariance form $A = \Xi^T \Xi$, where Ξ is a rectangular random matrix, play important roles, so that understanding properties of eigenvalue distributions of such matrices is useful.

Although the eigenvalue distribution of a random matrix is itself a random quantity, in many canonical cases it becomes deterministic in the limit of infinite dimensionality. This property, called the self-averaging property, allows us to analytically treat limiting eigenvalue distributions in the infinite-dimensionality limit.

A classic result regarding limiting eigenvalue distributions of matrices of the form $A = \Xi^T \Xi$ is that by Marčenko and Pastur [3]. Let Ξ be a $p \times N$ random matrix with independent and identically-distributed (i.i.d.) elements $\xi_{\mu i}$, $\mu = 1, \ldots, p$, $i = 1, \ldots, N$. Assume that $N^{1/2}\xi_{\mu i}$ is zero-mean, unit-variance, and finite higher-order moments. Then the limiting eigenvalue distribution $\rho_{\Xi^T\Xi}(\lambda)$ of the $N \times N$ random matrix $\Xi^T \Xi$ in the limit $N, p \to \infty$ while $\alpha = p/N$ kept finite is given by the so-called Marčenko-Pastur law

$$\lim_{N,\,p\to\infty} \rho_{\Xi^T\Xi}(\lambda) = \rho^{\mathrm{MP}}(\lambda) \equiv \begin{cases} \dfrac{\sqrt{4\alpha - (\lambda - 1 - \alpha)^2}}{2\pi\lambda} \chi_\alpha(\lambda) & (\alpha \geq 1) \\ (1-\alpha)\delta(\lambda) + \dfrac{\sqrt{4\alpha - (\lambda - 1 - \alpha)^2}}{2\pi\lambda} \chi_\alpha(\lambda) & (0 < \alpha < 1) \end{cases}$$

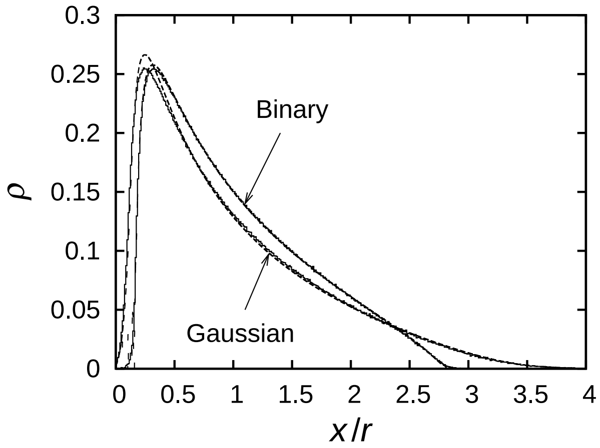
where

$$\chi_\alpha(\lambda) = \begin{cases} 1 & (\lambda \in [(1-\sqrt{\alpha})^2,\, (1+\sqrt{\alpha})^2]) \\ 0 & (\text{otherwise}) \end{cases}$$

is the indicator function of the interval $[(1-\sqrt{\alpha})^2,\, (1+\sqrt{\alpha})^2]$. The Marčenko-Pastur law is universal in the sense that it holds independent of details of the distribution of $\xi_{\mu i}$.

Ways out of the universality of the Marčenko-Pastor law are to consider distributions of $\xi_{\mu i}$ with fat tails, i.e., those which do not have higher-order moments, and to consider sparse random matrices. Motivated by problems in wideband wireless communication, we study the limiting eigenvalue distribution of $A = \Xi^T \Xi$ when Ξ is a sparse random matrix. Several random matrix ensembles are considered: The basic ones are defined by a rectangular random matrix Ξ with i.i.d. elements, each of which follows

$$p(\xi) = \left(1 - \frac{r}{N}\right) \delta(\xi) + \frac{r}{N} \pi(\xi),$$



(a) Poisson-weight ensembles with $r = 12$.

(b) Fixed-weight ensembles with row weight $K = 12$.

Figure 1: Limiting eigenvalue distributions of $A = \Xi^T\Xi$ with $\alpha = 0.3$. Nonzero elements take the values ± 1 with equal probabilities (Binary), or follow the standard Gaussian distribution (Gaussian). Histograms obtained numerically with $N = 6000$ (solid) and analytical results obtained via an approximation theory [4] (dashed) are shown.

where r is a parameter which corresponds to the average number of nonzero elements of Ξ per row, and where $\pi(\xi)$ is the distribution of the nonzero elements. Row and column weights (numbers of nonzero elements per row/column) are Poisson distributed in the basic ensembles in the infinite-dimensionality limit. We further consider ensembles which are defined by constraining the basic ones with row/column weight distributions. An example is a fixed-weight ensemble, in which row and column weights of Ξ are constrained to be prescribed constants.

Our basic finding is that the limiting eigenvalue distributions depend on the distribution $\pi(\xi)$ of nonzero elements, as well as the weight distributions, as demonstrated in Fig. 1. The figure also shows comparison between numerically obtained histograms and results obtained via an approximate analytical theory [4]. It can be observed that, even though there are notable discrepancies between them around the peaks and the tails, the analytical theory predicts the numerical results reasonably well.

Keywords

random matrix, limiting eigenvalue distribution, sparse matrix, Marčenko-Pastur law

References

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