Central limit theorem of random multiplicative processes and application to firm growth problems

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We consider the system which is consisted of N independent elements, whose time evolution is given by the following random multiplicative processes[1].

$$\begin{cases} x_1(t+1) = b_1(t)x_1(t) + f_1(t) \\ x_2(t+1) = b_2(t)x_2(t) + f_2(t) \\ x_3(t+1) = b_3(t)x_3(t) + f_3(t) \\ \qquad \vdots \\ x_N(t+1) = b_N(t)x_N(t) + f_N(t) \end{cases}$$

Here, $x_i(t)$ denotes the size of the i-th element in the time step t , $b_i(t) \geq 0$ is the growth rate of this element, and $f_i(t)$ is an independent noise. Both $b_i(t)$ and $f_i(t)$ are assumed to be independent random variables characterized by given distributions. We define the whole size of system, growth rate of the whole system and the noise term for the whole system as

$$X_N(t) \equiv \sum_{i=1}^{N} x_i(t) \qquad B_N(t) \equiv \frac{\sum_{i=1}^{N} b_i(t)x_i(t)}{\sum_{i=1}^{N} x_i(t)} \qquad F_N(t) \equiv \sum_{i=1}^{N} f_i(t) \quad ,$$

which satisfy the following time evolution for the aggregated quantities:

$$X_N(t+1) = B_N(t)X_N(t) + F_N(t)$$

Our numerical and theoretical analyses derived the following universal behaviors in the limit of large N.

The properties of aggregated growth rate B depend drastically on the condition of α, which satisfies $< b_i(t)^{\alpha} >= 1$. In the case of $0 < \alpha < 1$, the system growth rate distribution B obeys the distribution which depends on element's growth rate $b_i(t)$. In the case of $\alpha = 1$, B obeys the Cauchy distribution. In the case of $1 < \alpha < 2$, B obeys a stable distribution. And in the case of $\alpha \geq 2$, B obeys the normal distribution.

The statistical properties of the model are compared with the real firm data which covers practically all firms in Japan, about 1 million firms. It is shown that our simple model captures some of basic statistical properties[2]. Here, we consider that X_N corresponds to an annual sale of a company at the t-th year and $B_N(t)$ shows its growth rate. As shown in the following Figures the above simple model gives fairly good agreement with the real data.

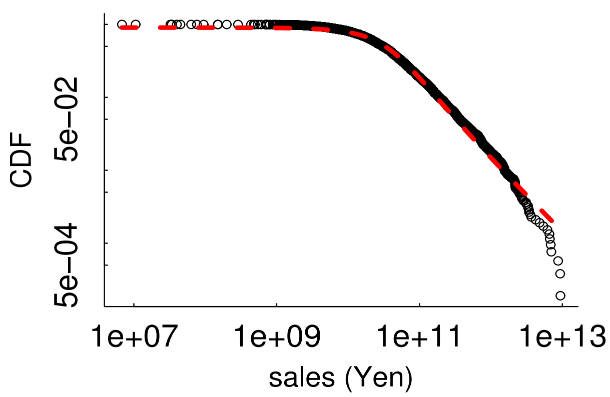

Fig. 1. The cumulative distribution function (CDF) of sales of Japanese firms with more than 500 employees. Black points show the CDF of sales from empirical data in 2003. Red dashed line shows the CDF of the stable distribution predicted by the model.

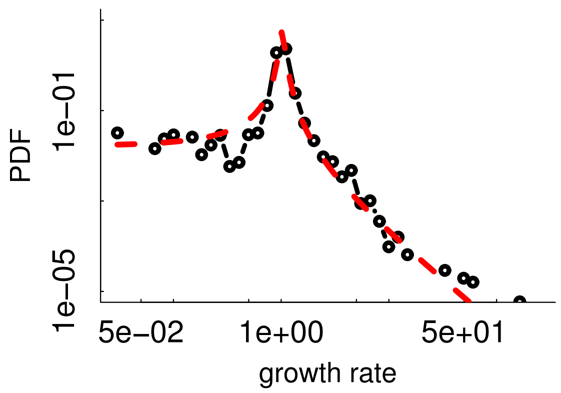

Fig. 2. The probability density function (PDF) of growth rates of Japanese firms with more than 500 employees. Black points show the PDF of growth rates from empirical data in 2003. Red dashed line shows the PDF of the Cauchy distribution predicted by the model.

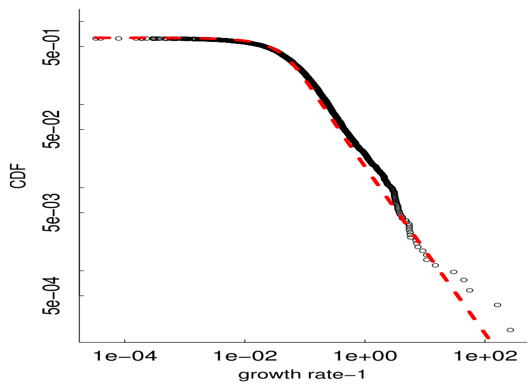

Fig. 3. The tail part of the CDF of growth rates of Japanese firms with more than 500 employees. Black points show the CDF of [growth rate-1], which means positive growth rates from empirical data in 2003. Red dashed line shows the PDF of the Cauchy distribution predicted by the model. Both distributions have power law tails with exponent -1.

Keywords

multiplicative process, firm growth, Zip's law, stable distribution, central limit theorem

References

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