# Duality and stationary distributions of wealth distribution models

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**Abstract**

We analyze a class of energy and wealth redistribution models, characterizing their stationary measures and showing that they have a discrete dual process. In particular we show that the wealth distribution model with non-zero saving propensity can never have invariant product measures.

We also introduce diffusion processes associated to the wealth distribution models by "instantaneous thermalization".

**Keyword:** Wealth distribution, energy models, redistribution models, duality, econophysics.

## 1 Intro

Wealth distribution models represent a flourishing field of the econophysical literature, see e.g. [2] for a recent overview. They represent simplified models of an economy, where at random instances agents exchange wealth, whereas the total wealth is conserved. These models are inspired from kinetic theory, since the exchange of wealth among economic agents is reminiscent of the exchange of energy in the collisions among the particles in a gas. Therefore, these models have striking similarities with the stochastic processes of energy exchange that are used as microscopic toy models of "the Fourier law" of heat conduction.

One particular aspect of wealth distribution models that makes them quite different from analogous models in mathematical physics, such as the KMP model for Fourier law [7], or the energy and particle transport models studied in [6], is the presence of the so-called "propensity to save", i.e. the tendency of agents to have only a fraction of their wealth involved in transaction events, while the other amount constitutes the savings, which are not transacted and allow for the accumulation of wealth.

It is therefore interesting to study wealth distribution models from the point of view of mathematical physics, using in particular the techniques of duality developed in [6] and [1]. The aim of this paper is to present first results in that direction, by restricting our attention to a general class of wealth distribution models on two sites.

For this class of models, we characterize their invariant measures and we prove that for non-zero propensity, there does not exist an invariant product measure, i.e., non-zero propensity necessarily leads to dependencies in the wealth of agents.

We also show that these models always have a discrete dual process, but the duality function factorizes if and only if the propensity to save is zero and the redistribution measure a beta distribution, corresponding to the unique case where we do have product invariant measures. In other words, factorization can only take place when the wealth distribution model is nothing but an energy redistribution model of the type described in [1]. The discrete dual process is easier than the original continuous process and can be used to study detailed time dependent properties of the wealth distribution.

We also introduce a class of diffusion processes naturally associated to the wealth distribution model, in the same way in which the KMP model is related to the Brownian energy process by instantaneous thermalization, see [6].

The two-agent models that we study here are to be considered as the building block of more complex models, because models with $N$ agents are always based on transactions between two agents, or put otherwise, in the language of kinetic theory, one restricts to binary collisions. From a microeconomic point of view, the transaction between two agents is the fundamental unit to study more general models of trade and exchange [8].

Moreover, the results that we obtain for two agents models remain partly true and contain relevant information for more complex $N$ agent models: the absence of product invariant measures for non-zero propensity, characterization of the redistribution laws (of beta-type) for which there are product measures in the zero propensity case, and the existence of a discrete dual process.

In the two agents case we can also completely characterize the set of stationary distributions, even in the presence of a non-zero propensity to save. The latter becomes more complex in models with more agents, and because complex dependencies will appear, it is not expected that for general models we can obtain analytic results. In forthcoming work we will however analyze the covariances among different agents in more complex $N$ agents models and show how they are related to the Green's function of the underlying random walk.

An important subject in the area of wealth distribution models is the emergence of power laws [3, 4, 5], and in particular of the Pareto law for the stationary wealth distribution marginals. In our models, Pareto laws only emerge for a particular choice of the "redistribution law". It is expected that in more complex models with $N$ agents and *annealed random propensities*, power laws can emerge due to the fact that with non-zero probability many agents have propensity close to one. Such a result is actually obtained in [9] for a particular model with deterministic redistribution law.

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